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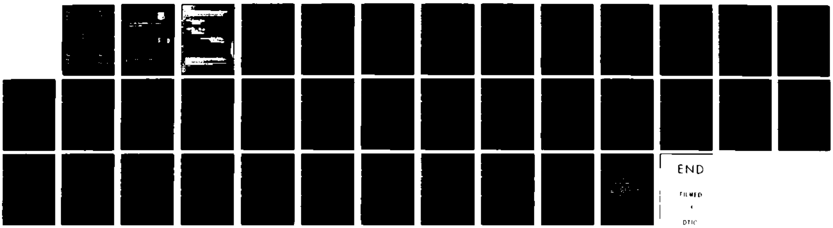

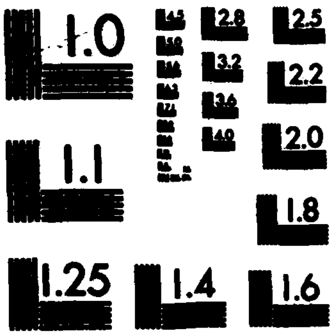

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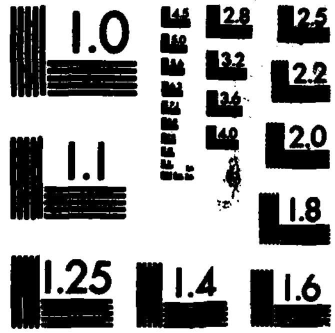

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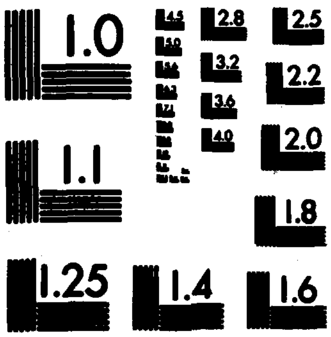

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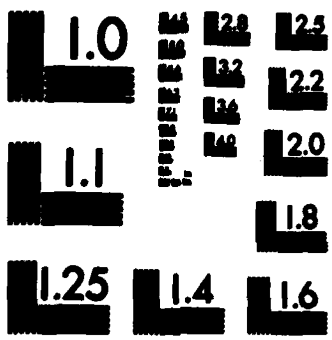

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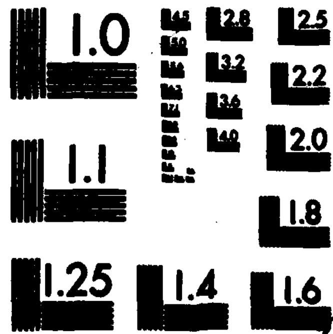

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July 1982

THE SINR PERFORMANCE OF CASCADED ADAPTIVE ARRAYS

The Ohio State University

Henry S. Ellts

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ROME AIR DEVELOPMENT CENTER
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This report analytically examines and discusses ways of maximizing the output signal-to-noise plus interference ratio (SINR) of cascaded processors. Investigations of various configurations of cascaded Power Inversion (PI) and Least Mean Square (LMS) processors are considered. The primary objective of the investigation is to derive techniques to overcome the limited dynamic range problem of the LMS processor. First a fully implemented cascaded array system of N-elements is considered where

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) each LMS array has N inputs, each of which is preceded by a PI array with each PI array connected to all antenna elements. This makes it possible to investigate the interaction of cascading the two processors, but it is an unrealistic implementation because of cost and complexity. Alternate approaches, referred to as thinned cascaded designs, examines the SINR output of cascaded arrays employing a minimum of PI control loops while maintaining maximum SINR. In general, it was determined that maximum SINR could not be maintained by arbitrarily thinned configurations, but could be achieved for a number of cases by careful selection of the PI steering vectors.

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CHAPTER I

INTRODUCTION

The least mean square (LMS) adaptive antenna array is used in communications systems for interference supression and desired signal tracking. It does this by adaptively weighting (both phase and magnitude) the signals from each antenna element to minimize (in the LMS sense) the difference between the array output and a reference signal. That is, it attempts to match the output with the reference. In doing this, it also provides the maximum signal to interference plus noise ratio (SINR) in the array output [1].

One of the problems with the LMS array is that it cannot adequately handle a large dynamic range of inputs. This problem is linked mathematically to the speed of response of the weights in the array. Ideally one would like to have the speed of response of the weights as fast as possible to provide rapid nulling of interference and good desired signal tracking. However, if the response speed of the array is too fast, the weights in attempting to match the output with the reference will begin to modulate the interfering signal with the reference signal. If this occurs the array will no longer distinguish between desired and undesired signals and will not suppress the interference. Unfortunately, the speed of response of the array is proportional to the power of the incident signals. Thus, the powers of the input signals cannot exceed a fixed level without these modulation effects. This effectively limits the dynamic range of signals that the array can handle. For most communications systems the dynamic range of the desired signals is not large enough to cause problems. However, in systems where some transmitters are close to the receiver and others are relatively far from the receiver, the dynamic range of desired signals may also be a problem.

To overcome the problem of limited dynamic range in the LMS array, it has been suggested [2] that each input of the LMS array be preceded by a power inversion (PI) array [3]. The PI array has no reference signal and for this reason is not as severely dynamic-range limited as the LMS array. Also, the PI array attenuates all signals which are above a threshhold value. The objective of this approach is to attenuate the powerful signals before they reach the LMS array and hence avoid the dynamic range problems. We refer to these systems as cascaded arrays.

In this report we examine the SINR of the cascaded array output. Specifically, we wish to know if preceding each input of the LMS array by a PI array destroys the LMS property of maximized SINR in the output. In Chapter III we show that under fairly liberal design constraints the steady state output of a "fully implemented" cascaded array is the same as the steady state output of a stand alone LMS array (with maximized SINR). A fully implemented N element cascaded array is one where the LMS array has N inputs, each of which is preceded by a PI array, and each PI array is connected to all of the N antenna elements.

There are good reasons why one would not build a fully implemented cascaded array. Cost and complexity are two factors which support arguments against a full implementation. An N element LMS array possesses N control loops if implemented at an intermediate frequency (as opposed to baseband). A corresponding N element fully implemented cascaded array possesses N^2+N control loops. Limiting the number of inputs to the PI arrays limits the number of weight control loops required, reducing both cost and complexity. Also, it may be desirable to limit the number of inputs to the PI arrays in order to purposely limit the degrees of freedom in these array. For lack of a better word, we call the process of reducing the inputs to the various arrays "thinning" the cascaded array.

In Chapters IV, V, and VI the output SINR of thinned cascaded arrays is examined. Chapter VI examines the output of cascaded arrays employing a minimum number of PI control loops. In general, we find that maximum SINR is not guaranteed for arbitrary thinned configurations, but can in many cases be guaranteed by careful selection of the PI steering vectors. The steering vectors determine the quiescent pattern of the PI arrays. The purpose of this report is to examine the output SINR of cascaded arrays and to present the designer with enough information so that it may be maximized.

CHAPTER II

BACKGROUND

Before discussing arrays in cascade we will examine a single (PI or LMS) adaptive array. Such an array is shown in Figure 1. The inputs to the array are the analytic signals[1] x_1 to x_N present at the respective antenna elements. These signals are each multiplied by a complex weight (w_1 to w_N) and then summed to yield the array output. By defining a vector input X

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \tag{1}$$

and a vector weight w

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \tag{2}$$

the output may be written as (with T denoting transpose)

$$\text{output} = X^T w = w^T X \quad . \tag{3}$$

In this report we will discuss two types of arrays, the LMS array and the PI array. The difference between the LMS array and the PI array is the feedback algorithm used to determine the weight vector. We first describe each of these arrays.

1. Analytic signal mathematics [4] is a method for treating two parameters of a signal, amplitude and phase, with one complex variable. It offers considerable simplification of the algebra involved in analysis and is used throughout most of this report.

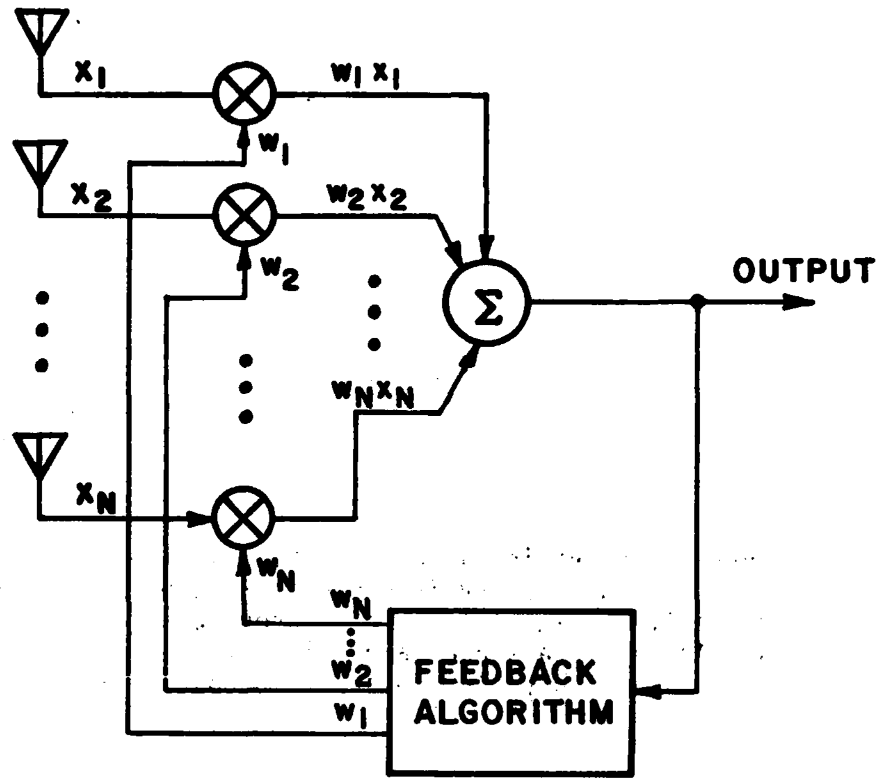


Figure 1. A generalized adaptive array.

II.A THE LEAST MEAN SQUARE ARRAY

In the LMS array the steady state weight vector minimizes a squared error cost function J_{LMS} given by

$$J_{LMS} = E[e^*e] \tag{4}$$

where $E[\cdot]$ is the expectation operator, the superscript * indicates complex conjugate, and e is the output error given by

$$e = r(t) - X^T w = r(t) - w^T X \quad . \tag{5}$$

Here r(t) is the analytic reference signal and w is the (not necessarily steady state) LMS weight vector. The value of w which minimizes J_{LMS} (i.e., the steady state weight vector) can be found analytically by differentiating J_{LMS} first with respect to the real part of the weight vector then with respect to the imaginary part of the weight vector and setting the two resulting equations equal to zero. Substituting

Equation (5) into Equation (4) and dropping the expressed dependence on t yields

$$J_{LMS} = E[r^*r] - S_X^\dagger w - w^\dagger S_X + w^\dagger \Phi_X w \qquad (6)$$

where $\dagger$ indicates conjugate transpose, S_X is the reference correlation vector and Φ_X is the input signal covariance matrix. S_X and Φ_X are defined by

$$S_X = E[X^*r] \qquad (7)$$

and

$$\Phi_X = E[X^*X^T] \quad . \qquad (8)$$

Equation (6) can be expanded by substituting for w

$$w = w_R + i w_I \qquad (9)$$

where

$$w_R = Re[w] \ , \qquad (10)$$

$$w_I = Im[w] \ , \qquad (11)$$

and

$$i = \sqrt{-1} \qquad (12)$$

This yields

$$\begin{aligned} J_{LMS} = {} & E[r^*r] - S_X^\dagger w_R - i S_X^\dagger w_I - w_R^T S_X + i w_I^T S_X \\ & + w_R^T \Phi_X w_R + w_I^T \Phi_X w_I - i w_I^T \Phi_X w_R + i w_R^T \Phi_X w_I \quad . \end{aligned} \qquad (13)$$

Differentiating Equation (13) with respect to w_R and setting the result equal to zero yields after some manipulation

$$Re[\Phi_X w] = Re[S_X] \quad . \qquad (14)$$

The same operation with respect to w_I yields

$$Im[\Phi_X w] = Im[S_X] \quad . \qquad (15)$$

Equations (14) and (15) can be combined to obtain a single complex equation

$$\Phi_X w = S_X \qquad (16)$$

The solution to Equation (16) is the weight vector which minimizes the cost function J_{LMS}. It is also the steady state LMS weight vector. If Φ_X is invertible the minimizing weight vector is given by[2]

$$w_L = \Phi_X^{-1} S_X \quad . \qquad (17)$$

2. Φ_X is invertible if there is noise present on each element and the noise is uncorrelated from element to element.

Here the subscript L denotes steady state LMS. This weight vector is known to yield maximum SINR in the LMS array output [1]. SINR is defined as the ratio of desired signal power to the sum of the noise power plus the interference powers. From Equation (3) the LMS output is given by

$$\text{output} = X^T \Phi_x^{-1} S_x \quad . \tag{18}$$

II.B THE POWER INVERSION ARRAY

The steady state weight vector w_{PI} for the PI array [3] is given by

$$w_{PI} = (I + k\Phi_x)^{-1} z \tag{19}$$

where k is a gain constant, I is the identity matrix, z is called the steering vector, and w_{PI} is the PI weight vector. If only noise is present at the antenna elements (with power σ^2 on each element and uncorrelated from element to element), $I+k\Phi x$ is diagonal with each term on the diagonal given by $1+k\sigma^2$. Under this condition Equation (19) reduces to

$$w_{PI} = \frac{1}{1+k\sigma^2} z \quad . \tag{20}$$

Thus, the vector z determines the array pattern under quiescent conditions -- hence the name steering vector.

In the rest of this report we consider adaptive array configurations that involve cascades of PI and LMS arrays. We begin with the fully implemented cascaded array.

CHAPTER III

THE FULLY IMPLEMENTED CASCADED ARRAY

We are investigating cascades of PI and LMS arrays in order to improve the dynamic range properties of adaptive arrays. This raises the question: How should these cascades be connected? For example, is it necessary that each LMS input be preceded by a PI array? Must each PI array be connected to all of the antenna elements? We would like to find a configuration that does not reduce the output SINR below that obtained by a single LMS array (which is the maximum obtainable SINR). Our purpose in this chapter is to consider first a fully implemented cascade. We will show that this configuration yields the maximum possible output SINR as long as the PI steering vectors are linearly independent. In later chapters we consider how the configurations can be thinned to reduce complexity without sacrificing output SINR.

We define an N element cascaded array to be fully implemented if the following requirements are met:

1. There are N PI arrays.

2. All of the antenna elements are connected to each of the PI arrays.

3. The LMS array has N inputs, each connected to the output of a PI array.

Figure 2 is a block diagram of an N-element fully implemented cascaded array. As in Equation (1) the input signal vector X is the vector of signals present at the terminals of each antenna element. We define an intermediate signal vector Y to be the vector of signals y_i

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \qquad (21)$$

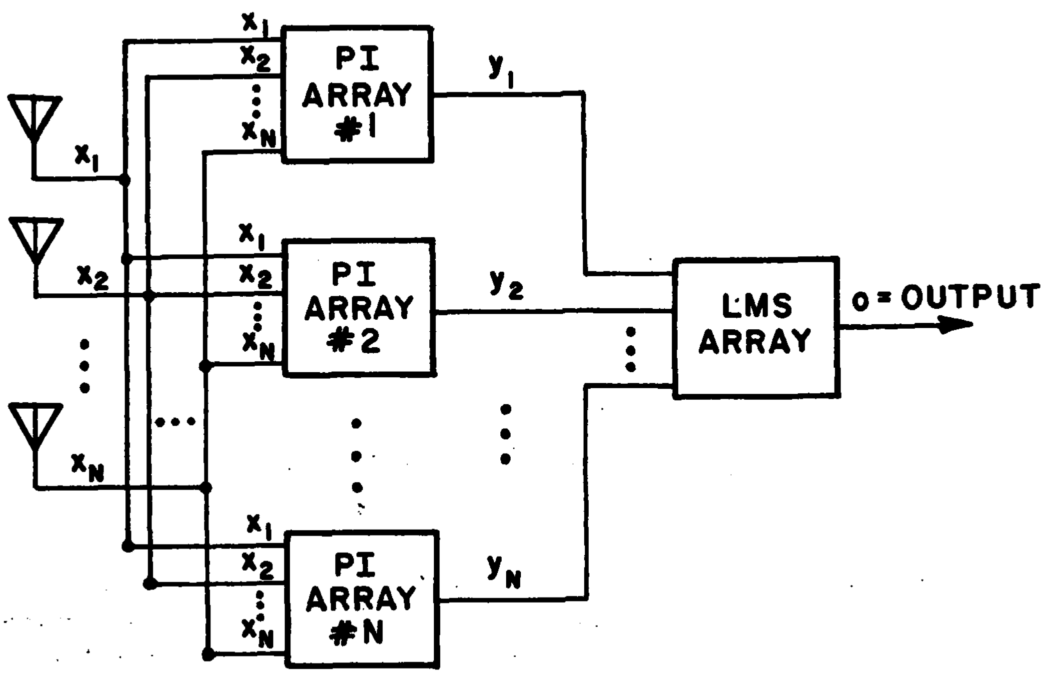


Figure 2. A fully implemented cascaded array.

where y_i is the output of the i^{th} PI array. Note from the block diagram that y_i is also the i^{th} input to the LMS array.

Since there are now several different power inversion arrays we adopt the following notation: let w_i be the steady state weight vector for the i^{th} PI array. The steady state LMS weight vector remains as w_L. To identify individual components of the weight vectors, double subscripts are used, where the first subscript refers to the component and the second subscript refers to the vector. That is,

$$w_i = \begin{bmatrix} w_{1i} \\ \vdots \\ w_{Ni} \end{bmatrix} \tag{22}$$

and

$$w_L = \begin{bmatrix} w_{1L} \\ \vdots \\ w_{NL} \end{bmatrix} . \tag{23}$$

It will also be convenient to define a PI weight matrix W where the columns of W are the PI weight vectors w_i:

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \tag{24}$$

With this notation, the components of the vector Y are given by

$$y_i = w_i^T X = X^T w_i \quad . \tag{25}$$

and the vector Y can be expressed as

$$Y = W^T X \quad . \tag{26}$$

From the discussion in Chapter II we know that the LMS array subject to an input vector Y will produce a weight vector w_L satisfying

$$\Phi_y w_L = S_y \tag{27}$$

where

$$\Phi_y = E[Y^* Y^T] \tag{28}$$

and

$$S_y = E[Y^* r] \quad . \tag{29}$$

By substituting Equation (26) for Y into Equations (28) and (29), Φ_y and S_y can be expressed as

$$\Phi_y = W^\dagger \Phi_x W \tag{30}$$

and

$$S_y = W^\dagger S_x \quad . \tag{31}$$

These can be substituted into Equation (27) to obtain

$$W^\dagger \Phi_x W w_L = W^\dagger S_x \quad . \tag{32}$$

Now, if W is invertible, the LMS weight vector is

$$w_L = W^{-1} \Phi_x^{-1} S_x \quad . \tag{33}$$

The output o from the cascaded array is then

$$o = Y^T w_L \quad . \tag{34}$$

Substituting Equations (26) and (33) into Equation (34) results in

$$o = X^T \Phi_x^{-1} S_x \tag{35}$$

We see that the output of the cascaded array as expressed in Equation (35) is exactly the same as the output from a single LMS array (Equation (18)). Thus, as long as the matrix W remains invertible, the output SINR of the cascaded array is maximized.

Since this result depends upon the invertibility of the PI weight matrix W, we now ask the question: What can be done to insure the invertibility of W? To answer this, note that W is invertible if the column vectors w_i (the columns of W) are linearly independent. That is, if there is no set of constants $\{b_i\}$ other than the null set for which

$$b_1 w_1 + b_2 w_2 + \ldots + b_N w_N = 0 \tag{36}$$

then the column vectors w_i are independent and W is invertible. Multiplying this equation by $(I+k\Phi_x)$ yields

$$b_1(I+k\Phi_x)w_1 + b_2(I+k\Phi_x)w_2 + \\ + \ldots + b_N(I+k\Phi_x)w_N = 0 \tag{37}$$

and from Equation (19)

$$b_1 z_1 + b_2 z_2 + \ldots + b_N z_N = 0 \tag{38}$$

where z_i is the steering vector for the i^{th} PI array. Equation (38) is the linear independence relation for the vectors z_i. Thus, the PI weight vectors w_i are linearly independent if the PI steering vectors z_i are linearly independent. This provides a convenient method for guaranteeing the invertibility of the matrix W. By defining a matrix Z whose columns are the steering vectors z_i,

$$Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_N \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix} \tag{39}$$

one can insure the linear independence of the vectors z_i and w_i by choosing the z_i such that the matrix Z is non-singular. Since the steering vectors z_i are parameters which are chosen by the array designer, it is a simple task to insure the invertibility of W.

Thus, we have shown that the fully implemented array yields maximum array output SINR as long as the steering vectors in the PI arrays are linearly independent. In the next chapters we consider how to thin the array without reducing SINR below its maximum value.

CHAPTER IV

THINNING BY REMOVAL OF PI ARRAYS

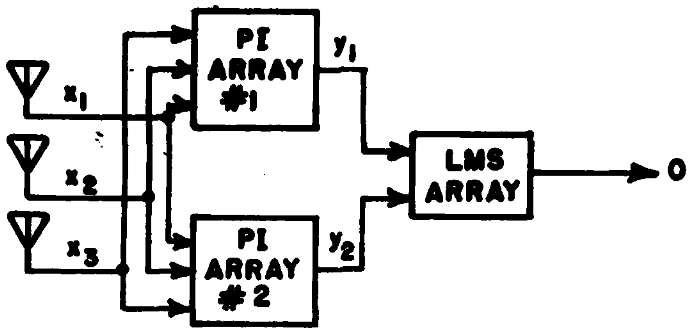
Consider for a moment the fully implemented cascaded array of Figure 2. There are two ways that this configuration can be thinned. One way is to remove inputs from one or more of the power inversion arrays. The second way is to remove inputs (and the corresponding PI arrays) from the LMS array. In this chapter we consider this latter method. We will show that eliminating PI arrays in this manner causes the output SINR to be reduced. Thinning via the first method will be considered in Chapters V and VI.

Suppose we are given an N-element fully implemented cascaded array as in Figure 2 and we remove the N^{th} PI array. We still retain N antenna elements, and each of the remaining PI arrays has N inputs as before, but the LMS array now has only N-1 inputs. Such a configuration is shown in Figure 3a for the case where N = 3. Recall that the matrix W of PI weight vectors was of dimension N x N for the fully implemented cascade. If a similar matrix W' is defined for this thinned configuration

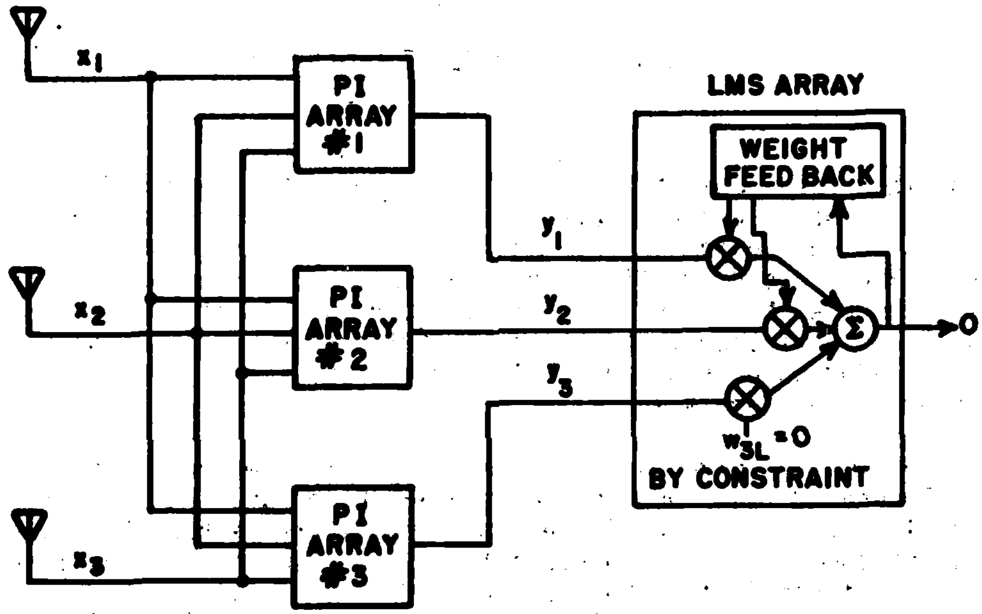
$$W' = \begin{bmatrix} w_1 & w_2 & \cdots & w_{N-1} \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix} \tag{40}$$

we observe that W' is of dimension N x N-1 and hence not invertible. Thus, the guarantee for maximum output SINR which was developed in Chapter III is not achieved. This does not imply that maximum SINR in the output is not possible -- only that it is not guaranteed.

The removal of the Nth PI array in a cascade is equivalent mathematically to a fully implemented cascaded array with the Nth component of the LMS weight vector w_L constrained to zero. For example, the cascaded array of Figure 3a (which has the 3rd PI array removed) is equivalent to the fully implemented cascaded array of Figure 3b (which has the 3rd component of w_L constrained to zero). The configuration of



3a.



3b.

Figure 3. A 3-element cascaded array with two PI arrays.
a. Physical situation. b. Constraint viewpoint.

Figure 3b is useful because it is easily analyzed with techniques of constrained minimization and because it permits the array behavior to be interpreted with geometrical arguments.

As described in Chapter II the steady state LMS weight vector minimizes a square error cost function

$$J_{LMS} = E[e^*e] = E[|e|^2] \quad . \tag{41}$$

There are some differences, however, between the case considered in Chapter II (a solitary LMS array) and the case we are considering here.

First, the input signal vector to the LMS array is now Y (instead of X), so the output error is

$$e = r - w^T Y = r - Y^T w \tag{42}$$

and the cost function J_{LMS} becomes (in a manner analogous to the derivation of Equation (6))

$$J_{LMS} = E[r^* r] - S_y^{\dagger} w - w^{\dagger} S_y + w^{\dagger} \Phi_y w \ . \tag{43}$$

Also, in this case the steady state LMS weight vector is the weight vector which minimizes J_{LMS} subject to the constraint that the Nth weight component is zero. Since we are working in complex weight space, both the real and imaginary parts of the N^{th} component are constrained. The previous case was one of unconstrained minimization.

There are several comments to be made about the cost function J_{LMS}. First, J_{LMS} is a Hermitian form[3] and is therefore real for all possible weight vectors w. Also, J_{LMS} has only one extremum and that point is a minimum since the error can be increased arbitrarily by increasing w. Finally, J_{LMS} as a function of the weight vector w is a quadratic hypersurface in a weight space of dimension 2N. If J_{LMS} is plotted versus the real or imaginary part of any two components of the weight vector w (i.e. versus any two of the 2N dimensions) a bowl shaped quadratic surface is obtained. The minimizing LMS weight vector w_L (unconstrained) is given by (in a manner analogous to the derivation of Equation (4))

$$w_L = \Phi_y^{-1} S_y \tag{44}$$

where Φ_y and S_y are given by Equations (28) and (29). The weight vector w_L results in the minimum possible value of the cost function J_{LMS} - the point at the bottom of the bowl. The shape of the quadratic hypersurface and the minimum value of J_{LMS} (J_{min}) are both functions of the intermediate signal vector Y (which in turn is a function of the input signal vector X). The minimum value of J_{LMS} can be calculated by substituting w_L (Equation (44)) for w in Equation (43) to obtain after some manipulation

$$J_{min} = E[r^* r] - S_y^{\dagger} \Phi_y S_y \tag{45}$$

By solving Equation (44) for S_y

3. The Hermitian form is the complex analog of the quadratic form of real space. If the problem is cast in real notation J_{LMS} becomes a quadratic form and we obtain a real weight vector with 2N components. The real and imaginary parts of the complex weight components each become individual components of the weight vector in real space.

$$S_y = \Phi_y w_L \tag{46}$$

and Equation (46) for $E[r^*r]$

$$E[r^*r] = J_{min} + S_y^\dagger \Phi_y S_y \tag{47}$$

and substituting these relations into Equation (44), one obtains

$$J_{LMS} = J_{min} + (w - w_L)^\dagger \Phi_y (w - w_L) \quad . \tag{48}$$

Equation (49) is useful because it clearly shows the Hermitian form of J_{LMS} and the penalty one pays for the use of weights other than w_L. Since Φ_y is a positive definite matrix, the product $(w - w_L)^\dagger \Phi_y (w - w_L)$ will always contribute positively to J_{LMS}. That is, the only way that J_{min} can equal J_{LMS} is if w equals w_L, forcing the product term to zero.

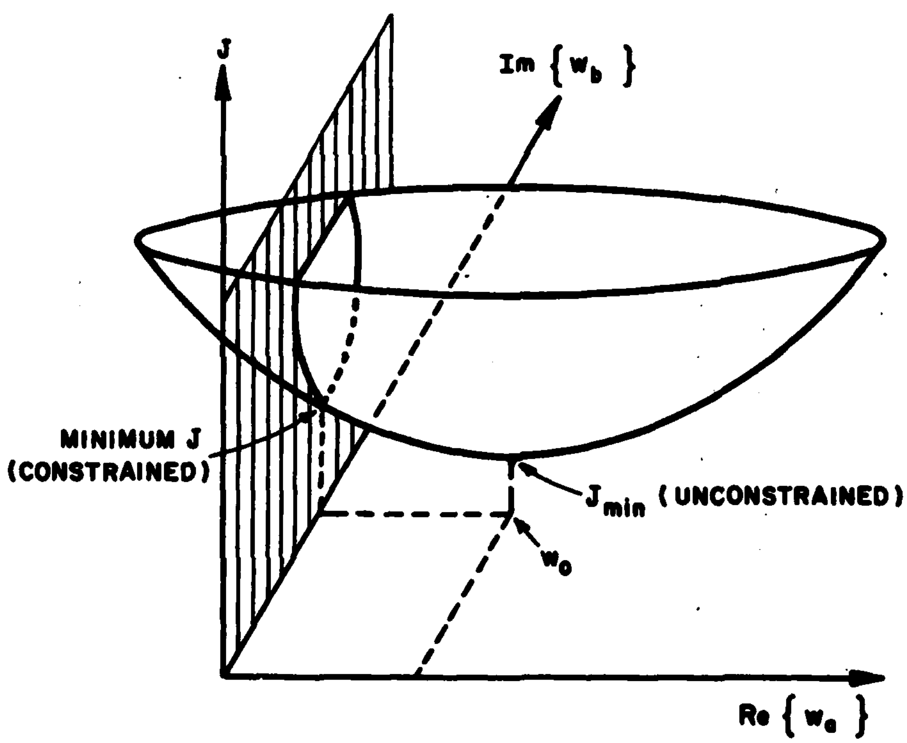


Figure 4. The LMS cost function with a constraint.

Figure 4 is an example which shows the cost function J_{LMS} and the effect of a single constraint. Here the cost function J_{LMS} is plotted versus $Re[w_a]$ and $Im[w_b]$, where w_a and w_b are two components of the complex weight vector w. In this example we constrain $Re[w_a]$ to zero. The shaded region in Figure 4 is the plane defined by the constraint equation

$$Re[w_a] = 0 \quad . \tag{49}$$

In this hypothetical case the weight vector will seek the minimum point of the intersection of the bowl and the constraint plane. From Equation (48) the constrained minimum will always be larger than the unconstrained minimum J_{min}. Physically, the larger value of J_{LMS} means that the array output does not match the reference as well as it would without the constraint. In other words, the array error signal has increased. For constraints of this type[4] the increased error is due to increased noise and interference in the output. This results in degraded output SINR.

Now consider the effect of constraining the N^{th} LMS weight component to zero by totally removing the N^{th} PI array in the cascade. This situation is similar to that of Figure 4 except now we have two constraints -- both the real and the imaginary parts of the weight component are constrained to zero. As in the example above, both the output error and the output SINR will increase. The conclusion of this argument is that one cannot remove PI arrays from a cascaded array without reducing the output SINR.

4. It is possible to increase the error without reducing the output SINR. This can be done by scaling all of the weight components by a common factor. This scales the output, causing a mismatch between the reference signal and increasing the error. The output SINR, however, remains the same. Changing only a single component (as in constraining a component to zero) will in general increase the error with a corresponding degradation in output SINR.

CHAPTER V

THINNING BY REMOVAL OF CONTROL LOOPS IN THE PI ARRAYS

Recall that we are discussing ways of thinning the cascaded array without degrading the array's output SINR. Chapter IV discussed the removal of control loops in the LMS array. In this chapter we discuss the removal of control loops in the PI arrays. Consider the N-element fully implemented cascade of Chapter III. Here, each PI array had all N antenna elements as inputs. One way of thinning this array would be to remove an input from one of the PI arrays. For example, in a three element cascaded array we could remove the third element from the third PI array. This results in the configuration of Figure 5. Along with the input that we remove, we also remove the corresponding weight control loop, including the weight vector component, the steering vector component, and all associated hardware. In this chapter and in Chapter VI we examine the effects of such configurations on the output SINR. That is, can the fully implemented cascaded array be thinned in this manner and still retain maximal SINR in the output? As in the fully implemented array of Chapter III, we will find that the answer to this question depends upon the choice of steering vectors used in the PI arrays.

To investigate thinning the cascaded array in this manner we again use the method of constraints. Removing the j^{th} input to a PI array is equivalent to constraining the j^{th} weight vector component to zero. Recall that maximum SINR for the fully implemented cascade is guaranteed if the matrix W is invertible. This will be true for these configurations also. The constraints force certain elements of W to zero. For the example of Figure 5 the matrix W is

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & 0 \end{bmatrix} \quad (50)$$

The elements which are forced to zero by the constraints correspond to the control loops removed from the PI arrays. As in the fully

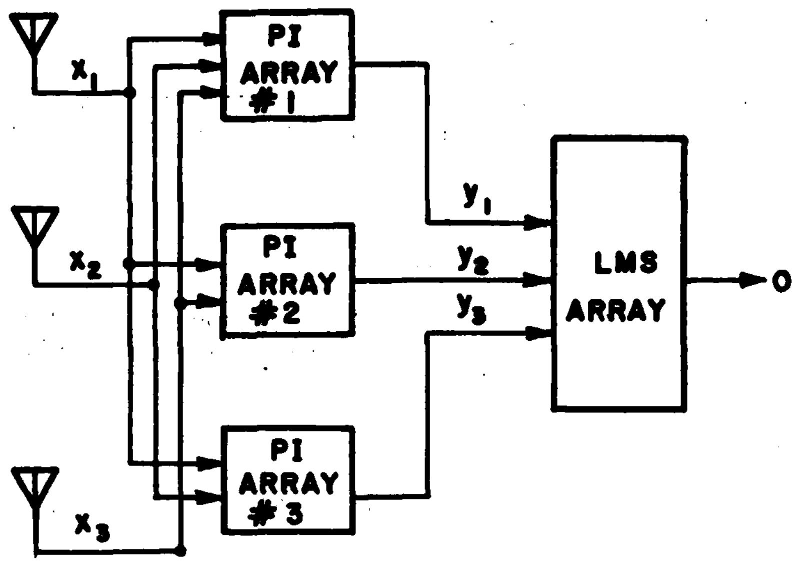


Figure 5. A cascaded array with a thinned power inversion array.

implemented case, the invertibility of W is not obvious by inspection. However, in many cases the invertibility of W can be determined by examining the steering vectors z_i. This will be done in this chapter. Before proceeding further, however, we pause to develop the mathematics of the PI array with constraints.

Consider a 3-element PI array. From Equation (19) the PI weight vector is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{3i} \end{bmatrix} = \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{bmatrix} \tag{51}$$

where the a_{jk} are the elements of the matrix $(I+k\Phi_x)$. Suppose the third input and control loop is now removed, so that the array is reduced to a 2-element array. The weight vector is then given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix} = \begin{bmatrix} z_{1i} \\ z_{2i} \end{bmatrix} \tag{52}$$

The components of w_i are found by Cramer's rule to be

$$w_{1i} = \frac{a_{22}z_{1i} - a_{12}z_{2i}}{a_{11}a_{22} - a_{21}a_{12}} \tag{53}$$

and

$$w_{2i} = \frac{a_{11}z_{2i} - a_{21}z_{1i}}{a_{11}a_{22} - a_{21}a_{12}} \quad . \tag{54}$$

This answer is also obtained by solving the original system (Equation (51)) with the constraint that $w_{3i} = 0$. Equation (51) then becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ 0 \end{bmatrix} = \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i}' \end{bmatrix}, \tag{55}$$

where the prime on z_{3i} indicates that this component may differ from the component z_{3i} in Equation (51). For this equation to have a solution (i.e. to be consistent) z_{3i}' must be given by

$$z_{3i}' = a_{31}w_{1i} + a_{32}w_{2i} \tag{56}$$

where w_{1i} and w_{2i} are given by Equations (53) and (54). When these are substituted, z_{3i}' is found to be

$$z_{3i}' = \frac{a_{31}(a_{22}z_{1i} - a_{12}z_{2i}) + a_{32}(a_{11}z_{2i} - a_{21}z_{1i})}{a_{11}a_{22} - a_{21}a_{12}} \quad . \tag{57}$$

Equation (57) shows that the value which z_{3i}' must have in order to force w_{3i} to zero is dependent upon the input signal environment. This is observable in the dependence upon the a_{jk}. Thus, unlike the other steering vector components, z_{3i}' is not a constant. Should the input environment change, z_{3i}' must also change to maintain w_{3i} at zero. For lack of a better work, we call this component a "floating" component of the steering vector.

In a manner analogous to the previous developments, one can write

$$(I+k\Phi_x)w_i = z_i' \tag{58}$$

where w_i has components constrained to zero and z_i' has corresponding floating components. By defining a steering vector matrix Z' with columns composed of the steering vectors z_i',

$$Z' = \begin{bmatrix} z_1' & z_2' & \cdots & z_N' \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \tag{59}$$

Equation (58) can be extended to

$$(I+k\Phi_x)W = Z' \tag{60}$$

where W now has constrained components and Z' has floating components. Since $(I+k\Phi_x)$ is non-singular, W is invertible if and only if Z' is invertible. This is the fundamental result of this chapter. It permits us to examine the invertibility of W by examining the matrix Z'. In the next chapter we use this result to examine several configurations of the cascaded array.

CHAPTER VI

SPECIAL CONFIGURATIONS

In the previous chapter a mathematical base was developed for examining thinned configurations of the cascaded array. In this chapter we apply these and other techniques to some special configurations which may be of interest. We have two goals. First, we would like to find the minimum configuration which still maximizes the output SINR. By minimum configuration, we mean the configuration with the least number of PI control loops. This configuration is important because it has minimum cost and complexity. Secondly, we wish to present a broad class of configurations which yield maximum output SINR, although they are not necessarily minimum.

VI.A THE MINIMUM CONFIGURATION

The simplest thinned array is one where only one element is connected to each PI array as in Figure 6. Normally, when one speaks of arrays, one implies that there is more than one element. The PI arrays of Figure 6 are single element "arrays". We choose to begin with this case as a tutorial example, however, because it is the simplest case. We show here that this configuration gives maximum output SINR.

To see this, we examine the PI weight matrix W. For this configuration, W is a diagonal matrix and is given by

$$W = \begin{bmatrix} w_{11} & & & \\ & w_{22} & & \\ & & \cdot & \\ & & & w_{N,N} \end{bmatrix} \qquad (61)$$

Its determinant will be non-zero if each of the diagonal elements are non-zero. Each of these in turn is given by the scalar equation (from

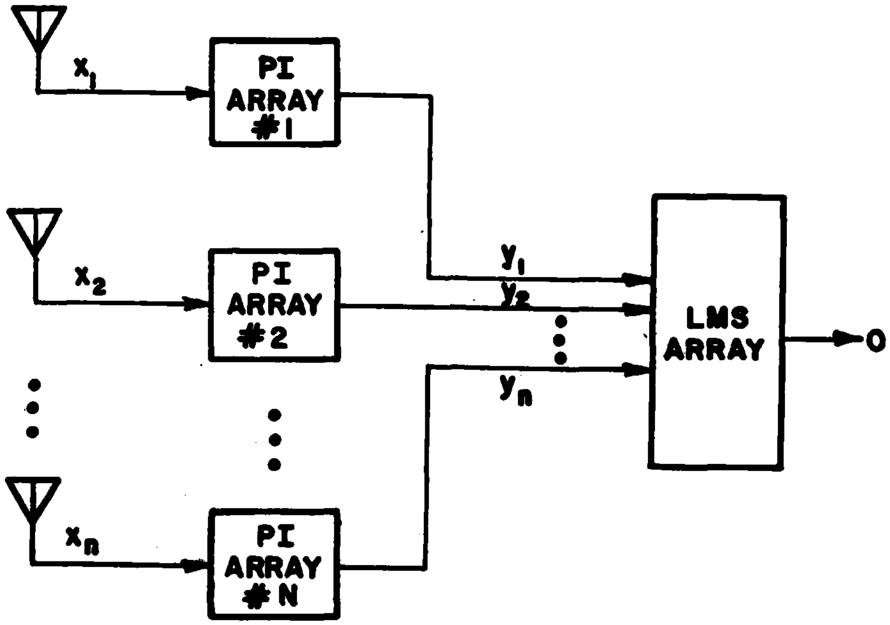


Figure 6. A cascaded array with one element PI "arrays"

Equation (19))

$$w_{ii} = z_{ii}/(1+k\phi_{ii}) \tag{62}$$

where $\phi_{ii} = E[x_i^* x_i]$. Since ϕ_{ii} can never go to infinity, one can insure w_{ii} is never zero by choosing the steering vector component z_{ii} non-zero. If this is done for each PI control loop, the matrix W is non-singular and the configuration is guaranteed to yield maximized SINR in the LMS output.

Although the configuration of Figure 6 will maximize SINR, one would probably not build such an array. The reason is that none of the PI "arrays" have the capability of nulling an interfering signal. An N-element array has the capability of nulling N-1 signals. If one were designing a cascaded array for an environment of interfering signals, one would probably put enough elements in the PI arrays to provide for nulling of the interference. For example, if one interfering signal were expected, a cascaded array composed of two element PI arrays would be desired. The question would then become: Can a configuration be found using two element PI arrays which has maximum output SINR? One such configuration is shown in Figure 7. This configuration was found by trial and error. We show below that with proper choice of PI steering vectors it also yields maximum output SINR. We will see later that the configuration is not unique. There are other equivalent configurations.

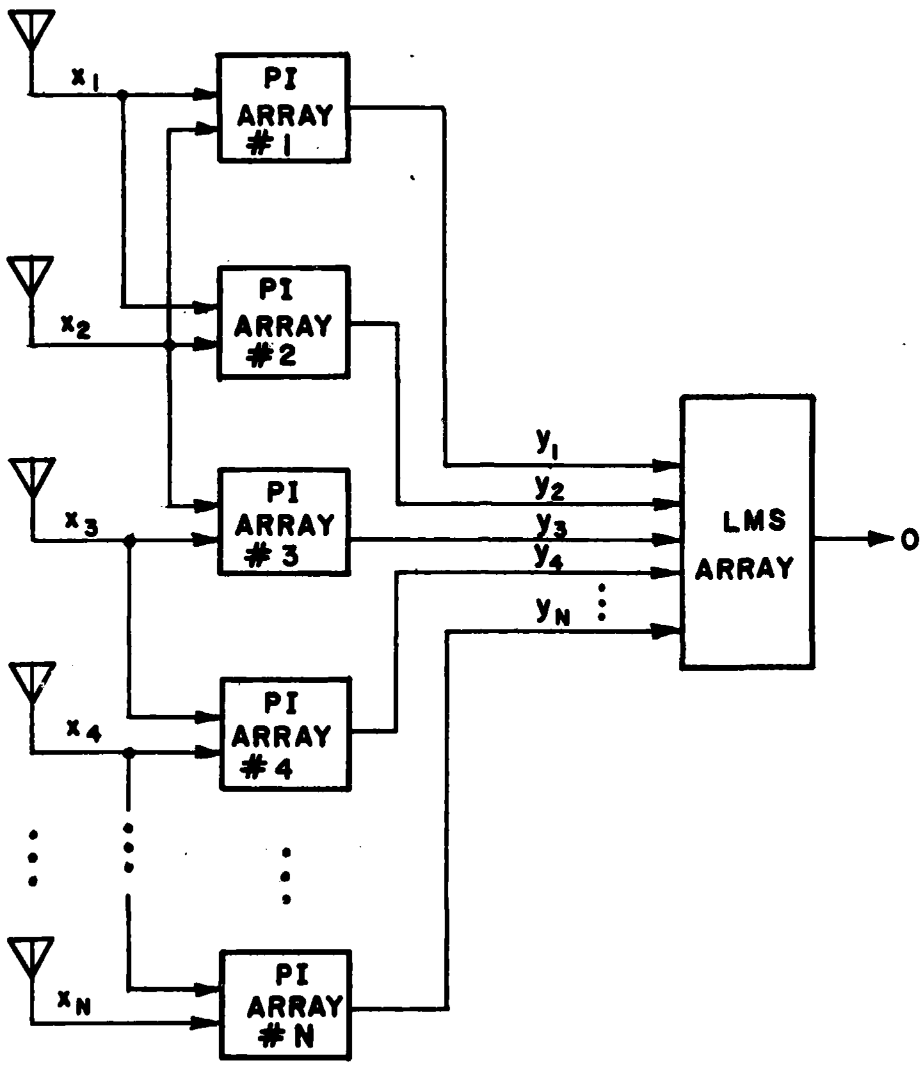


Figure 7. A cascaded array with two element PI arrays.

As in the previous case, we examine the output SINR of this configuration by examining the invertibility of the matrix W, or equivalently, the linear independence of the columns of W. For the array of Figure 7, W is given by

$$W = \begin{bmatrix} w_{11} & w_{12} & & & & \\ w_{21} & w_{22} & w_{23} & & & \\ & & w_{33} & w_{34} & & \\ & & & w_{44} & \cdot & \\ & & & & \cdot & w_{N-1,N} \\ & & & & & w_{N,N} \end{bmatrix} \tag{63}$$

We will argue along the following lines. First, we will show that two conditions are sufficient for guaranteeing the linear independence of the columns of W. These are (1.) the first two columns of W must be linearly independent and (2.) all diagonal elements in the remaining columns must not go to zero. We will then show how a designer can insure that the conditions are met.

To see that the two conditions are sufficient for the linear independence of the columns of W, suppose that both conditions are true. Note that the first two columns of W do not have a third component. Since they do not have a third component, no weighted sum of these columns can produce the third column, which has a non-zero third component. Mathematically speaking, there is no set $\{b_1, b_2\}$ for which

$$b_1 \begin{bmatrix} w_{11} \\ w_{21} \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} w_{12} \\ w_{22} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ w_{23} \\ w_{33} \end{bmatrix} \quad . \tag{64}$$

Thus, all three columns are linearly independent. Similarly, no weighted sum of the first three columns can produce the fourth column, and so on. Therefore, given the independence of the first two columns and non-zero diagonal elements in the remaining columns, all of the columns of W are linearly independent.

We now show how a designer can guarantee that the first condition (i.e., the linear independence of the first two columns) is met. The weight components of the first two PI arrays satisfy the equation

$$(I+k\Phi_x)'W'' = Z'' \tag{65}$$

where $(I+k\Phi_x)'$ is the 2 x 2 upper left block of $(I+k\Phi_x)$,

$$W'' = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{66}$$

and

$$Z'' = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \tag{67}$$

Note that $(I+k\Phi_x)'$ is a Hermitian, positive definite, non-singular matrix. Therefore, W" is invertible and its columns linearly independent if Z" is invertible. Since the choice of steering vectors is a designer's option, Z" can be chosen invertible, guaranteeing the independence of the columns of W" . The columns of W" when augmented with zeros are the first two columns of W. Hence, by choosing Z" invertible, the independence of first two columns of W is guaranteed.

The second condition (insuring that the remaining diagonal elements never vanish) can also be met by proper choice of the steering vectors. In any PI array it is possible to prevent any given weight component from becoming zero. This can be done by choosing the steering vector component which corresponds to the given weight component to be non-zero and choosing all other steering vector components (those that correspond to existing control loops) to be zero. A proof of this statement is given in Appendix I. Thus, it is possible for a designer to guarantee that the remaining diagonal components of W never vanish.

We have shown above that the invertibility of the matrix W and hence maximum output SINR of this cascaded array configuration is guaranteed by proper choice of the steering vector components. We also remarked earlier that the configuration is not unique. We now show this. Suppose we take the configuration of Figure 7 and we renumber the antenna elements by interchanging the labels of elements one and three. That is, we call the signal from the first element x_3 and the signal from the third element x_1. The actual circuitry is not changed, so the output SINR remains maximized.

The new weight matrix for this configuration (denoted by primes) is

$$W' = \begin{bmatrix} 0 & 0 & w_{13}' & w_{14}' & & & \\ w_{21}' & w_{22}' & w_{23}' & 0 & & & \\ w_{31}' & w_{32}' & 0 & 0 & & & \\ 0 & 0 & 0 & w_{44}' & w_{45}' & & \\ & & & & w_{55}' & & \\ & & & & & \ddots & w_{N-1,N}' \\ & & & & & & w_{N,N}' \end{bmatrix} \qquad (68)$$

or

$$W' = \begin{bmatrix} 0 & 0 & w_{33} & w_{34} & & & \\ w_{21} & w_{22} & w_{23} & 0 & & & \\ w_{11} & w_{12} & 0 & 0 & & & \\ 0 & 0 & 0 & w_{44} & w_{45} & & \\ & & & & w_{55} & & \\ & & & & & \ddots & w_{N-1,N} \\ & & & & & & w_{N,N} \end{bmatrix} \qquad (69)$$

where the w_{ij} (unprimed) are elements of the matrix W (Equation (63)). An examination of W' shows that is is identical to W except that the first and third rows have been interchanged. In a similar manner the outputs of the PI arrays could be renumbered. This would result in a weight matrix with columns interchanged. The net result of this is that any row or column interchange of the matrix W is equivalent to renumbering the inputs or outputs of the PI arrays.

Thus, the invertibility of any weight matrix W which can be permuted to the form of Equation (63) by row and column interchanges can be guaranteed by the proper choice of steering vectors. The matrices which fit this class have the following features:

1. Each of the N columns possess two weight components.

2. Each of the rows except two possess two weight components.

3. One row has one weight component.

4. One row has three weight components.

This defines a class of cascaded arrays which are "minimum" configurations and which maximize SINR in the output. These arrays have the following characteristics (corresponding to the above characteristics of the weight matrix):

1. Each PI array has two inputs.

2. Each antenna element except two is an input to two PI arrays.

3. One antenna element is an input to only one PI array.

4. One antenna element is an input to three PI arrays.

VI.B GENERAL CONFIGURATIONS

The minimum configurations of Part VI.A, composed of two element PI arrays, have the capability of nulling one strong jammer. It may be desirable to incorporate multi-element (more than two) PI arrays in the cascaded array to provide for the simultaneous nulling of more than one jammer. It is possible to do this without resorting to the fully implemented cascaded array of Chapter III. This section presents one possible way of doing this while still maintaining maximized SINR in the array output. Our purpose is not to endorse this configuration (although it is reasonable for the scenario given above) but rather to present the methods used to examine it.

Recall that in Chapter V we obtained the relation

$$(I+k\Phi_X)W = Z' \qquad (70)$$

where Z' is the matrix of steering vector components with floating components. Since $(I+k\Phi_X)$ is always invertible, W is invertible if Z' is invertible. One problem which occurs is the unpredictability of the floating component terms in the matrix Z'. These terms are dependent upon the input signal scenario and may (for certain input conditions) assume values which result in a singular Z'.

Consider the special case where Z' is upper triangular

$$Z' = \begin{bmatrix} z_{11} & * & * & \cdots & * \\ 0 & z_{22} & * & & * \\ 0 & 0 & z_{33} & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & z_{N,N} \end{bmatrix} \tag{71}$$

with each diagonal element non-zero. The determinant of this matrix is the product of the diagonal elements and is therefore non-zero. This configuration is interesting because it permits the removal of PI control loops which correspond to the superdiagonal elements of Z' (the asterisks of Equation (71)). This introduces floating component terms into these elements, but the determinant remains non-zero and hence the matrix remains invertible. Control loops may not be removed from positions which correspond to diagonal or subdiagonal elements of Z'. This would introduce unpredictable floating component terms into the diagonal or subdiagonal elements and render the invertibility of Z' unpredictable. As in the previous case, row and column interchanges may be used to transform a given matrix into triangular form in order to show invertibility.

An example of a cascaded array which fits this matrix is the array[5] of Figure 8. This array has the weight matrix

$$W = \begin{bmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{22} & 0 & 0 \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix} \tag{72}$$

and the matrix of steering vectors

$$Z' = \begin{bmatrix} z_{11} & * & * & * \\ z_{21} & z_{22} & * & * \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} \tag{73}$$

Here the asterisks represent floating steering vector components. By choosing

$$z_{11} = z_{22} = z_{33} = z_{44} = 1 \tag{74}$$

and

$$z_{21} = z_{31} = z_{41} = z_{32} = z_{42} = z_{43} = 0 \tag{75}$$

5. This array incorporates PI arrays with different degrees of freedom. One PI array can null three jammers, one can null two jammers, and the remaining two PI arrays can each null a single jammer.

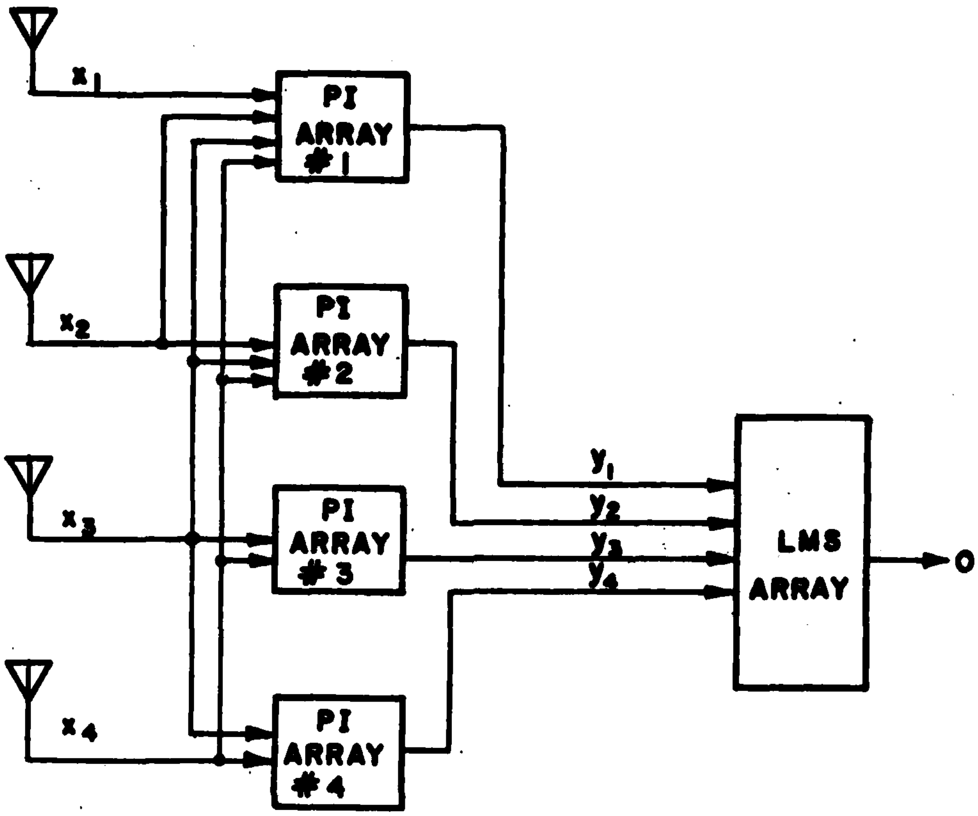


Figure 8. A cascaded array derived from an upper triangular matrix Z'.

the matrix Z' becomes

$$Z' = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{76}$$

By choosing the steering vector components in this way, the matrix Z' is non-singular regardless of the values of the unspecified components (z_{34} and the floating components (asterisks). Thus, the array of Figure 8 produces maximal SINR. However, the configuration is not the important issue here. The important point is the triangular form of the matrix Z'. This form allows one to investigate a great number of configurations for maximum output SINR.

CHAPTER VII

CONCLUSIONS

The cascaded adaptive arrays discussed in this report are composed of an LMS processor with each LMS input preceded by a power inversion (PI) array. It is a characteristic of the LMS array that the speed of response of the LMS weights are proportional to the input signal power. Powerful signals can cause the LMS array to respond too fast, resulting in undesirable modulation effects in the output. The objective of cascading adaptive arrays is to use the inversion characteristics of the PI arrays to limit the input power to the LMS section, thereby eliminating these modulation effects.

The LMS array (when used by itself) possesses the property of maximizing the steady state output SINR. Preceding each LMS input with a PI array may or may not destroy this property. This report presents several techniques for guaranteeing maximum SINR in the output of a cascaded array. In a fully implemented cascaded array each PI array is connected to the full set of antenna elements. For this configuration maximized output SINR is guaranteed by choosing the steering vectors linearly independent. It is also possible to have maximum output SINR from cascaded arrays where each PI array is connected to only a subset of the antenna elements. However, this can be accomplished only through further restrictions on the PI steering vectors.

Maximized output SINR is a desirable property. With the techniques presented in this report it is possible to choose array configurations which have this property. In addition to the output SINR, other factors should be considered in choosing a cascaded array configuration. These include transient behavior, weight jitter, and dynamic range of the signals and weights throughout the system. These topics are either the subject of current research or are proposed for future examination.

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APPENDIX I

INSURING A NON-ZERO WEIGHT COMPONENT IN PI ARRAYS

In Chapter V we stated that is is always possible to prevent a weight component w_{ji} of a PI weight vector w_i from going to zero. This is shown below.

As noted previously, there are two viewpoints for considering the removal of PI control loops. One is the constraint viewpoint, where the removed control loops are imagined to exist but with their weight components constrained to zero. The other viewpoint treats the thinned PI array as a standard PI problem but with reduced dimensionality. For example, if we start with an N element PI array and remove an input and corresponding control loop an N-1 element PI array remains. In this appendix we adopt this second viewpoint.

Consider the PI weight equation

$$(I+k\Phi_X)w_i = z_i \tag{A-1}$$

where $(I+k\Phi_X)$, w_i, and z_i are perhaps of reduced dimensionality. This equation may be solved for any component of w_i by the use of Cramer's rule. This yields an equation of the form

$$w_{ji} = \mathrm{Det}(I+k\Phi_X)'/\mathrm{Det}(I+k\Phi_X) \tag{A-2}$$

where $(I+k\Phi_X)'$ is the matrix $(I+k\Phi_X)$ with its j^{th} column replaced by the steering vector z_i. The numerator of Equation (A-2) can be expanded about the column vector z_i to yield

$$w_{ji} = (-1)^{j+1}[z_{1i}\Delta_1 - z_{2i}\Delta_2 + \ldots]/\mathrm{Det}(I+k\Phi_X) \tag{A-3}$$

where Δ_k are subdeterminants of $(I+k\Phi_X)$. Note the following facts:

1. $(I+k\Phi_x)$ is a positive definite matrix so $Det(I+k\Phi_x) > 0$.

2. With the exception of Δ_j, each of the other Δ_k are determinants of non-Hermitian matrices which may or may not be positive definite.

3. Δ_j is the determinant of a Hermitian submatrix of $(I+k\Phi_x)$ which is positive definite.

Since Δ_j is greater than zero, one way of guaranteeing that w_{ji} never goes to zero is to pick z_{ji} non-zero and all other components of z_i equal to zero.

To see this, consider the following example. Suppose we have a four element PI array and we wish to insure that w_{2i}, the second component of the weight vector w_i, never goes to zero. The PI weight equation is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{3i} \\ w_{4i} \end{bmatrix} = \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \\ z_{4i} \end{bmatrix} \qquad \text{(A-4)}$$

where the a_{jk} are the elements of $(I+k\Phi_x)$. Solving Equation (A-4) for the second component of w_i with the use of Cramer's rule results in

$$w_{2i} = \frac{\begin{vmatrix} a_{11} & z_{1i} & a_{13} & a_{14} \\ a_{21} & z_{2i} & a_{23} & a_{24} \\ a_{31} & z_{3i} & a_{33} & a_{34} \\ a_{41} & z_{4i} & a_{43} & a_{44} \end{vmatrix}}{Det(I+k\Phi_x)} \qquad \text{(A-5)}$$

Since the denominator is the determinant of a positive definite matrix, it can never go to zero. Therefore, insuring that w_{2i} never vanishes amounts to insuring that the numerator of Equation (A-5) never vanishes. Expanding the numerator about the second column results in

$$w_{2i} = \frac{- z_{1i}\Delta_1 + z_{2i}\Delta_2 - z_{3i}\Delta_3 + z_{4i}\Delta_4}{Det(I+k\Phi_x)} \qquad \text{(A-6)}$$

where the subdeterminants Δ_k are given by

$$\Delta_1 = \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \qquad \text{(A-7)}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \qquad \Delta_4 = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix} \tag{A-7}$$

Notice that of all the Δ_k, only Δ_2 is the determinant of a Hermitian submatrix. Δ_2 is, in fact, the determinant of a 3 x 3 submatrix of the form $(I+k\Phi_x)$, and so is non-zero. Thus, to guarantee that w_{2i} never vanishes, we choose z_{2i} to be non-zero and all the other components of the vector z to be zero.

This same phenomenon occurs for the general case. In general, to insure that a component w_{ji} of a weight vector w_i never goes to zero, pick z_{ji} non-zero and all components of the steering vector z_i zero.